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APPROXIMATE FORMULAS FOR DEFLECTION OF COMPRESSED FLEXIBLE BARS

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The exact relationship between a load that compresses longitudinally a flexible elastic hinge-supported bar and the deflection was expressed by Euler in the form of a complete elliptic integral of the first kind [1–3]. The equilibrium postbuckling modes of compressed bars were first studied by Lagrange and investigated in great detail using the tables of elliptic integrals in [2]. Also, there is a great variety of approximate formulas expressing deflection via loading, but these are applicable from an engineering viewpoint only for loads that exceed a critical load by not more than 10%.

In the present paper, an approximate formula is proposed which makes it possible to compute deflections on simple calculators with a relative error not greater than 3% up to loads that exceed a critical (Euler's load) by a factor of 3.5. It is shown that some small simplifications in the obtained formula yield a rougher formula which is more accurate than many of the well-known approximate formulas and allows a more compete qualitative description of load–deflection dependences for any loads.

1. Statement of the Problem. Exact Solution. We consider a flexible hinge-supported bar loaded by an axial compressive force P which preserves its magnitude and direction upon deformation of the bar. We assume that the length L of the axial line of the bar is unchanged and the bar axis is bendable only in the plane. We consider the load–deflection dependence for an equilibrium configuration that branches from the unbuckled mode after the first critical load $P_* = EI(\pi/L)^2$ (EI is the bending stiffness of the bar). Note that the load–deflection dependence for higher equilibrium modes which correspond to the subsequent critical loads is obtained from the previous dependence by a simple transformation of the scale [4, 5]. Let us construct an approximate formula that allows us to calculate, with an error not greater than 3%, bar deflections under loads that are severalfold greater than the critical load. Let us compare with one another and with the exact solution the various available approximate formulas for bar deflections as a function of load.

The exact dependence of bar deflection on longitudinal load is given in parametric form [2–5]:

$$f = \frac{a}{L} = \frac{2k}{\pi\sqrt{\lambda}}, \quad \lambda = \frac{P}{P_*} = \left(\frac{2}{\pi}\int\limits_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right)^2, \tag{1.1}$$

where the parameter $0 \leqslant k \leqslant 1$ has the geometric sense of the sine of half the angle between the tangent to the bent bar axis at its vertex and the initial rectilinear direction of the bar; $1 \leqslant \lambda$ is a dimensionless load parameter; and a and $f < 1/2$ are the maximum dimensional and dimensionless bar deflections, respectively.

2. Approximate Formulas. Approximate formulas are frequently [2–6] obtained from exact formulas (1.1) by expanding the complete elliptic integral of the first kind in a power series in the small parameter k:

$$\int\limits_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \ldots\right). \tag{2.1}$$

Another method of obtaining approximate formulas is that of using various approximations (linearizations) of the assumed nonlinear differential equation of the equilibrium of the bar's elastic line including those with

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TABLE 1

Formula load	Deflection λ 1.01	1.1	1.7489	2.1834	4	Formula load	Deflection λ 1.01	1.1	1.7489	2.1834	4
	f						f				
(1.1)	0.088974	0.2543	0.4031	0.392	0.313	(2.6)	0.0896	0.271			
(2.2)	0.0900	0.285				(2.7)	0.0895	0.268			
(2.3)	0.0899	0.281				(2.8)	0.08898	0.2557	0.448	0.463	0.441
(2.4)	0.0895	0.266	0.405	0.24		(2.9)	0.088974	0.2543	0.4019	0.389	0.301
(2.5)	0.08896	0.251	0.09			(2.10)	0.0891	0.259	0.445	0.449	0.390

refined boundary conditions taking into account the longitudinal shift of the base upon deflection of the bar [1, 7–10]. Using this method von Mises [7, p. 436] derived the formula

$$f = (2\sqrt{2}/\pi)\sqrt{\lambda - 1}, \tag{2.2}$$

which relates the dimensionless load λ to the maximum dimensionless bar deflection f. Formula (2.2) has been most widely used particularly after it was derived by various methods [3, 4, 8, 11, 12]. Note that it was obtained in [3, p. 443; 4, p. 32] by approximating the solution by truncated series, i.e., by the first method, and in [11, p. 62] by the Koiter technique; a similar formula is given in [12, p. 245] despite inaccurate use of catastrophe theory [13]. The formula

$$f = (2\sqrt{2}/\pi)\sqrt{\lambda - 1}[1 - (1/8)(\lambda - 1)], \tag{2.3}$$

which is somewhat more exact than (2.2), is given in [1, p. 74]; it is obtained in [7 and 8, p. 37] just as (2.2), i.e., by the second method. Using the first method (by means of series) Krylov [2, p. 501] obtained an approximate formula which can be written in our notation as

$$f = (2\sqrt{2}/\pi)\sqrt{\lambda - 1}[1 - (41/64)(\lambda - 1)]. \tag{2.4}$$

Nikolai [3, p. 443] criticizes the von Mises approximate analysis which was used to obtain formula (2.3) and to derive, by means of series, the formula

$$f = (2\sqrt{2}/\pi)\sqrt{\lambda - 1}[1 - (19/16)(\lambda - 1)]. \tag{2.5}$$

Note that (2.3)–(2.5) differ from one another only by the coefficient of $\lambda - 1$ and hence, the graphs of these functions are qualitatively the same. As for (2.5), series were used in [5, p. 137] to construct the formulas

$$f = (2\sqrt{2}/\pi)\sqrt{1 - 1/\lambda}; \tag{2.6}$$

$$f = (4/\pi)\sqrt{\sqrt{1/\lambda} - 1/\lambda}. \tag{2.7}$$

Formula (2.6) can also be obtained by the second method [9, p. 429] using refined boundary conditions that take into account the shift of the base. However, an attempt [10, p. 489] to derive (2.7) by the second method was not successful: it resulted in a formula that yields deflections about half as large as those obtained from the exact solution. Finally, in [6, p. 23] use was made of series to obtain the formula

$$f = (4\sqrt{2}/3\pi\sqrt{\lambda})\sqrt{\sqrt{9\sqrt{\lambda} - 8} - 1}, \tag{2.8}$$

which is more exact than the formulas above, and from which, in the same paper, formula (2.7) was derived via simplification.

The same method, taking into account terms in expansion (2.1) up to k^6 inclusive, can be used to construct the formula

$$f = (4/\sqrt{3}\pi\lambda)\sqrt{\sqrt{6\lambda - 2} - 2}. \tag{2.9}$$

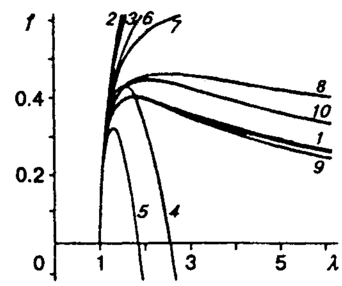

Fig. 1

Assuming that $\lambda - 1 \ll 1$ and using the approximation

$$\sqrt{\sqrt{6\lambda-2}-2} = \sqrt{2}\sqrt{\sqrt{1+(3/2)(\lambda-1)}-1} \approx \sqrt{2}\sqrt{1+(3/4)(\lambda-1)-(9/32)(\lambda-1)^2-1}$$
$$\approx \sqrt{2}\sqrt{(3/4)(\lambda-1)[1-(3/8)(\lambda-1)]} \approx \sqrt{3/2}\,\sqrt{\lambda-1},$$

we write a simplified version of (2.9) in the form

$$f = (2\sqrt{2}/\pi\lambda)\sqrt{\lambda-1}. \tag{2.10}$$

3. Comparison of the Formulas. Conclusions. Calculating the deflection by formulas (2.7)–(2.10), we obtain $f \to 0$ for $\lambda \to \infty$, which corresponds to a mechanical sense, in contrast to $f \to \infty$ for $\lambda \to \infty$ for (2.2), $f \to -\infty$ for (2.3)–(2.5), and $f \to 2\sqrt{2}/\pi$ for (2.6).

Table 1 presents deflection values $f = a/L$ for a load $\lambda = P/P_*$, which are calculated by solution of system (1.1) with an assigned accuracy and in accordance with approximate formulas (2.2)–(2.10). From the exact solution it can be found that the bar deflection reaches a maximum $f_{\max} \approx 0.403$ for a load $\lambda \approx 1.75$, and the bar ends converge for $\lambda \approx 2.18$, i.e., the bar forms a loop [2]. Deflections for these key load values of practical interest are also given in Table 1. The empty spaces of Table 1 correspond to numbers that do not have a physical meaning, for instance, $f < 0$ or $f > 0.5$.

Figure 1 shows a graph of the exact load-deflection dependence (curve 1) and graphs constructed by formulas (2.2)–(2.10) to which correspond curves 2–10. The data in Table 1 and the graphs show convincingly formulas (2.2)–(2.7) to be applicable only to loads that exceed the critical load by not more than 10% ($\lambda = 1.1$). More exact formulas (2.8)–(2.10) were constructed by approximation of the exact solution [1–6] of the assumed nonlinear equation. But if this nonlinear equation is first replaced by an approximate (linearized) equation and is then solved exactly, rougher formulas (2.2) and (2.6) are obtained. In addition, it can be seen that (2.10) is mainly more exact than (2.8).

Among approximate formulas (2.2)–(2.10) formula (2.9) is the most exact one; its maximum relative error for $1 \leqslant \lambda \leqslant 4$ equals $\approx 3.8\,\%$ and is reached for $\lambda = 4$. An increase in λ increases the error of formula (2.9), but (2.9), (2.8), and (2.10) adequately reflect qualitatively the load–deflection dependence for any loads. The maximum of function (2.9) is reached for $\lambda \approx 1.732$ and is equal to 0.402, which differs only slightly from the exact value of $f = 0.403$ for $\lambda \approx 1.749$. For (2.8) and (2.10) we have, respectively, $f \approx 0.465$ for $\lambda \approx 2.39$ and $f \approx 0.45$ for $\lambda = 2$.

Note that formula (2.9) yields understated deflection values, while (2.10), overstated values. These formulas can be used for tentative rough estimates in more complicated problems in which bars are composite elements of a construction. Formulas (2.9) and (2.10) may also prove useful in instrument making, for instance, in calculating mechanical regulators.

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